Cofactor expansion

Recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det A = ad - bc$. We showed that A is invertible if and only if $\det A \neq 0$.

Our goal in this section is to define the determinant for arbitrary matrices, and get an analogous condition for invertibility.

We define the determinant *inductively*.

3×3 matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Define

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

A w/ 1st row and column removed

A w/ 1st row, 2nd column removed

A w/ 1st row, 3rd column removed

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

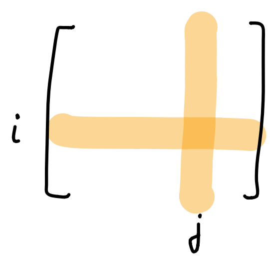
Ex: $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ 0 & 5 & 4 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix}$

$$= 1(0-5) - 3(-4-0) + 2(-5-0)$$
$$= -5 + 12 - 10 = -3$$

To define the determinant for $n \times n$ matrices, we need to define cofactors:

Def: Let A be an $n \times n$ matrix, and let

$A_{ij} = (n-1) \times (n-1)$ matrix obtained from A be deleting row i, column j.



The (i,j)-cofactor of A is defined

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

$$(-1)^{i+j} = \begin{cases} 1 & \text{if } i+j \text{ even} \\ -1 & \text{if } i+j \text{ odd} \end{cases}.$$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & & \\ \vdots & \vdots & & \ddots & \end{bmatrix}$$

This is called the sign of the (i,j)-position.

Def: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A).$$

row 1 entries

This is called the <u>cofactor expansion</u> of det A along row 1.

Ex:

$$\begin{vmatrix} 1 & 3 & -2 \\ 5 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} = \underset{+}{(-1)^{1+1}\,1 \begin{vmatrix} 0 & 6 \\ 0 & 1 \end{vmatrix}} + \underset{-}{(-1)^{1+2}\,3 \begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix}} + \underset{+}{(-1)^{1+3}(-2) \begin{vmatrix} 5 & 0 \\ 0 & 0 \end{vmatrix}}$$

$$= 1 \cdot 0 - 3 \cdot 5 - 2 \cdot 0 = -15$$

Thm: The determinant of a matrix can be computed by using the cofactor expansion along <u>any</u> row or column.

Ex: We can calculate the determinant of the above matrix along column 2:

$$\begin{vmatrix} 1 & 3 & -2 \\ 5 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} = (-1)^{1+2}\,3 \begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix} + (-1)^{2+2}\,0 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + (-1)^{2+3}\,0 \begin{vmatrix} 1 & -2 \\ 5 & 6 \end{vmatrix}$$

$$= -3 \cdot 5 = -15$$

or row 3:

$$\begin{vmatrix} 1 & 3 & -2 \\ 5 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} = 0 - 0 + 1 \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = 0 - 15 = -15$$

One nice thing about the above theorem is that we can sometimes choose a column or row containing mostly zeros to do the cofactor expansion along.

Ex:

$$\begin{vmatrix} 1 & 5 & 7 & 9 \\ 0 & 3 & 0 & 0 \\ -2 & 9 & 1 & 1 \\ 0 & -5 & 2 & 0 \end{vmatrix} = -0 + 3\begin{vmatrix} 1 & 7 & 9 \\ -2 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} - 0 + 0$$

$$= 3\left(0 - 2\begin{vmatrix} 1 & 9 \\ -2 & 1 \end{vmatrix} + 0\right)$$

$$= 3\left(-2\left(1-(-18)\right)\right)$$

$$= 3(-2(19)) = 3(-38) = -114$$

The following properties also help us find the determinant of large matrices:

Properties of the determinant:

Let A be an $n \times n$ matrix.

1.) If A has a row or column of zeros, then $\det A = 0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 6 & 7 \end{vmatrix} = 0$$

2.) If two distinct rows or columns of A are interchanged, the determinant of the resulting matrix is $-\det A$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -\begin{vmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix}$$

3.) If a row or column of A is multiplied by a constant u, the determinant of the resulting matrix is $u\det(A)$.

$$-2\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 5 \\ 7 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 2 & -10 \\ 7 & 0 & -2 \end{vmatrix}$$

4.) If two distinct rows or columns of A are identical, then $\det A = 0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

5.) If a multiple of one row of A is added to a different row (or if a multiple of a column is added to a different column), the det of the resulting matrix is $\det A$, i.e. it doesn't affect the determinant.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

We can use these properties to make determinant calculations easier:

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$. What is $\det A$?

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 3 \\ 0 & 1 & 0 \\ 5 & 3 & -1 \end{vmatrix} = -0 + 1 \begin{vmatrix} 3 & 3 \\ 5 & -1 \end{vmatrix} - 0 = -3 - 15$$

$$= -18$$

adding column 2 to column 1 doesn't change determinant

expansion along row 2

Ex: If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 6$, compute $\det A$,

where $A = \begin{bmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{bmatrix}$

$$\det A = (-1)(3)\det \begin{bmatrix} a+x & b+y & c+z \\ x & y & z \\ p & q & r \end{bmatrix}$$

(pulling out scalar multiples)

$$= -3 \det \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$$

(subtracting row 2 from row 1)

$$= (-3)(-1)\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

(switching rows 2 and 3)

$$= (-3)(-1)(6) = 18$$

Ex: What are the determinants of elementary matrices?

Type I: switch two rows of I = identity matrix

$$\det E = -\det I = -1$$

Type II: k times a row of I

$$\det E = k \det I = k$$

Type III: row i + multiple of row j

$$\det E = \det I = 1$$

Ex: Let $A = \begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix}$.

For which values of x is $\det A = 0$?

$$\det A = \begin{vmatrix} 0 & 1-x & 1-x^2 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = \begin{vmatrix} 0 & 1-x & 1-x^2 \\ 0 & x-1 & 1-x \\ 1 & 1 & x \end{vmatrix} = 1 \begin{vmatrix} 1-x & 1-x^2 \\ x-1 & 1-x \end{vmatrix}$$

row 1 $- x$ (row 3)

row 2 $-$ row 3

$$= (1-x)(1-x) - (x-1)(1-x^2)$$
$$= (1-x)^2 + (1-x)(1-x)(1+x)$$
$$= (1-x)^2(1 + (1+x))$$
$$= (1-x)^2(2+x)$$

This is zero if $x=1$ or $x=-2$.

Ex:
$$\det \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} = a \begin{vmatrix} e & f & g \\ 0 & h & i \\ 0 & 0 & j \end{vmatrix} = ae \begin{vmatrix} h & i \\ 0 & j \end{vmatrix} = aehj$$

Called an upper triangular matrix (zeros below diagonal)

$aehj$ = product of diagonal

Thm: If A is an upper triangular matrix (0 below diagonal) or a lower triangular matrix (0 above diagonal), then $\det A$ is the product of the entries along the diagonal.

Practice problems: 3.1: 1cfjkm, 5d, 6, 7, 13, 15